SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2001

MATHEMATICS

EXTENSION 1

Time allowed:	*2 Hours*
	(plus five minutes reading time)
Examiner:	*E.Choy*

DIRECTIONS TO CANDIDATES

- *ALL* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Start each question on a new answer sheet.
- Additional answer sheets may be obtained from the supervisor upon request.

NOTE: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate examination paper for this subject.

Question 1. (12 marks)

(a) Find the acute angle (correct to the nearest minute) between the lines $3x + 2y = 7$ and $4x - 3y = 2$. **2**

(b) Using the expansion of $\tan(\alpha - \beta)$, or otherwise, show that $\tan(-15^0) = \sqrt{3} - 2$. **2**

(c) Find $\lim\limits_{x \to 0} \dfrac{\sin 4x + \tan x}{x}$. **2**

(d) Differentiate with respect to *x*: **2**

(i) $y = \ln(\cos x)$

(ii) $y = \tan^{-1} 3x$

(e) Solve $2\cos^2 x + 3\sin x - 3 = 0$, where $0 \le x \le 2\pi$. **2**

(f) Find the co-ordinates of the point *P* that divides the interval joining the points $A(-3,4)$ and $B(-1,0)$ externally in the ratio 4:3. **2**

Question 2. (12 marks)

(a) Find the general solution of $\tan x = \sqrt{3}$. Give your answer in a concise, general form. **2**

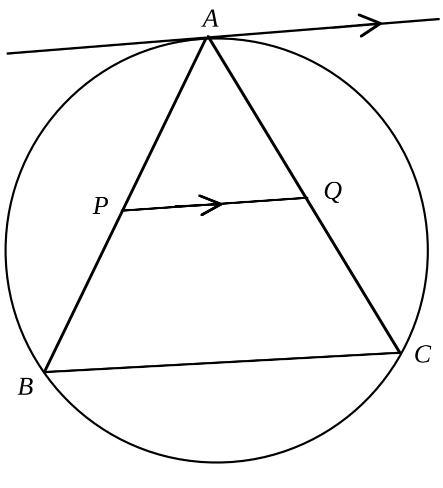
(b) How many different 9-letter "words" can be made from the letters of *ISOSCELES*? **2**

(c) Find the domain and range of the function $y = \sin^{-1}\left(1 - \sqrt{x}\right)$. **1**

(d) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{3 - x^2}}$. **2**

(e) Find all solutions to $\dfrac{x}{x^2 - 1} > 0$. **2**

(f) Given $AB = AC$, and that the tangent at A is parallel to PQ. **3**



Prove:

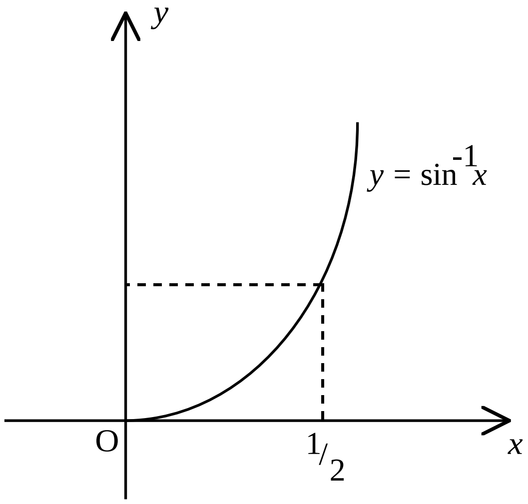
(i) $AP = AQ$

(ii) BC is parallel to the tangent at A.

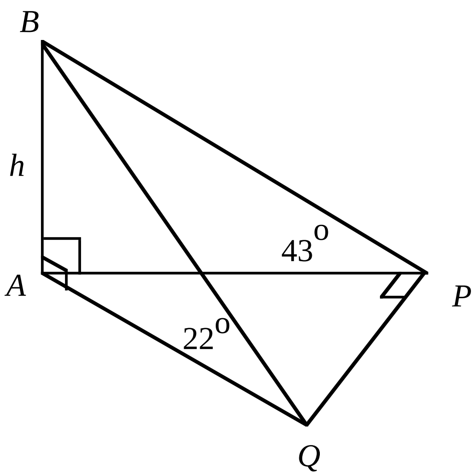
(iii) $PCBQ$ is a cyclic quadrilateral.

Question 3. (12 marks)

(a) Find the exact area bounded by the curve $y = \sin^{-1} x$, the x-axis, and the ordinate $x = \frac{1}{2}$ as shown in the diagram. **4**



(b) **4**



The elevation of the top of a hill (B) from a place P due east of it is 43^0, and from a place Q, due south of P, it is 22^0. The distance from P to Q is 400m. If h is the height of the hill, show that

$$h^2 = \frac{160000}{\cot^2 22 - \cot^2 43}.$$

(c) Find $\int \sec^2 x . \tan^2 x \, dx$ using the substitution $u = \tan x$. **4**

Question 4. (12 marks)

(a) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. **8**

(i) Find the co-ordinates of *A*, the point of intersection of the tangents to the parabola at *P* and *Q*.
(You may use the fact that equation of the tangent to the parabola $x^2 = 4ay$ at the point $T(2at, at^2)$ is $y = tx - at^2$.)

(ii) Suppose further that *A* lies on the line containing the focal chord which is perpendicular to the axis of the parabola.

(α) Show that $pq = 1$.

(β) Show that the chord *PQ* meets the axis of the parabola on the directrix.

(b) If $y = x^3 - 2x^2 + 3$ **4**

(i) find the equation of the tangent to the curve at (2, 3), and

(ii) find the point at which the tangent meets the curve again.

Question 5. (12 marks)

(a) Prove by mathematical induction that for positive integral n, $3^{3n} + 2^{n+2}$ is divisible by 5. **4**

(b) By considering the function $f(x) = x^3 - 7$, use one step of Newton's method to find a better approximation to $\sqrt[3]{7}$ than 2. Leave your answer in exact fractional form. **3**

(c) The speed v m/s of a point moving along the x-axis is given by $v^2 = 90 - 12x - 6x^2$, where x m is the displacement of the point from the origin. **3**

(i) Prove that the motion is simple harmonic.

(ii) Find the period, the centre of motion, and the amplitude.

(d) (i) Prove that $\cos 2\theta = \dfrac{1 - x^2}{1 + x^2}$, where $x = \tan\theta$. **2**

(ii) Use the above result to deduce that $\tan\frac{\pi}{8} = \sqrt{2} - 1$.

Question 6. (12 marks)

(a) Given $y = \sin^{-1}(\cos x)$: **4**

(i) Find $\dfrac{dy}{dx}$.

(ii) Evaluate $y = \sin^{-1}(\cos x)$ if $x = \pi$.

(iii) Sketch $y = \sin^{-1}(\cos x)$ for $-\pi \le x \le \pi$.

(b) Whilst playing tennis, Eric serves a ball from a height of 1 8 metres. If he hits the ball in a horizontal direction at a speed of 35 m/s, find (using $g = 10\,\text{ms}^{-2}$): **6**

(i) How long before the ball hits the ground.

(ii) How far the ball will travel before bouncing.

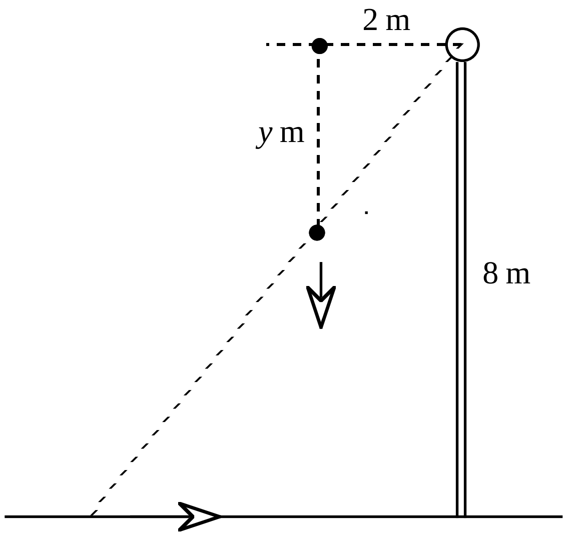
(iii) By how much the ball clears the net, which is 0 95 m high and 14 metres distant.

(c) (i) Find $\dfrac{d}{dx}\left(xe^{x}\right)$. **2**

(ii) Use the result in Part (i) to evaluate $\displaystyle\int_0^1 xe^{x}\,dx$

Question 7. (12 marks)

(a) A street lamp is 8 m high. A small object 2 m away from the lamp falls vertically downward. **6**



(i) Show that when the object has fallen y metres, the shadow it casts on the horizontal ground is $\dfrac{16}{y}$ metres from the base of the lamp.

(ii) When the object has fallen 6 m, it is travelling at 10 m/s. At what speed is its shadow moving?

(iii) At what height does the object have the same speed as its shadow?

(b) A function $f(x)$ is defined by the rule $f(x) = \left(e^x - 1\right)\ln x$ for $0 < x \leq 1$. **6**

(i) Evaluate $f'(1)$.

(ii) Using the fact that $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$, show that $f'(x) \to -\infty$ as $x \to 0$.

(iii) Hence or otherwise show that $f(x)$ has a stationary value, and determine its nature.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

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Mathematics Extension 1

Sample Solutions

Q1 a) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

gradients $m_1 = -\frac{3}{2}$, $m_2 = \frac{4}{3}$

2

$\tan\theta = \frac{-\frac{3}{2} - \frac{4}{3}}{1 + -\frac{3}{2}\times\frac{4}{3}}$

$= \frac{17}{6}$

$\theta = 70^\circ 34'$

(b) $\tan(30-45) = \frac{\tan 30 - \tan 45}{1 + \tan 30 \tan 45}$

2

$= \frac{\frac{\sqrt{3}}{3} - 1}{1 + \frac{\sqrt{3}}{3}\times 1} = \frac{\sqrt{3}-3}{3+\sqrt{3}}$

$= \frac{6\sqrt{3}-12}{6} = \sqrt{3}-2$

(c) $\lim_{x\to 0} \frac{\sin 4x}{x} + \lim_{x\to 0} \frac{\tan x}{x}$

$4 \lim_{x\to 0} \frac{\sin 4x}{4x} + \lim_{x\to 0} \frac{\tan x}{x}$

2

$= 5.$

(d) (i) $y = \ln(\cos x)$

$\frac{dy}{dx} = \frac{1}{\cos x} \times \frac{-\sin x}{1}$

1

$= -\tan x$

(ii) $y = \tan^{-1} 3x$

1

$\frac{dy}{dx} = \frac{3}{1+9x^2}$

(e) $2(1-\sin^2 x) + 3\sin x - 3 = 0$

$2 - 2\sin^2 x + 3\sin x - 3 = 0$

$2\sin^2 x - 3\sin x + 1 = 0$

2

$(2\sin x - 1)(\sin x - 1) = 0$

$\sin x = \frac{1}{2}$ $\sin x = 1$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\frac{\pi}{2}$

(f) $k = 4$, $l = -3$

$(5, 12)$ 2

Question 2

(a) $\tan x = \sqrt{3}$

$\therefore x = 180n + \tan^{-1}(\sqrt{3})$

$\boxed{x = 180n + 60^\circ}$ or $\boxed{x = n\pi + \frac{\pi}{3}}$

b) $\dfrac{9!}{2!\,3!}$ = $\begin{cases} \text{repetition of S } (\times 3) \\ \text{repetition of E } (\times 2) \end{cases}$

$\boxed{30240}$

c) domain: $-1 \le 1 - \sqrt{x} \le 1$

$\therefore -2 \le -\sqrt{x} \le 0$

$\therefore 0 \le \sqrt{x} \le 2$

$\therefore \boxed{0 \le x \le 4}$ $(\frac{1}{2})$

range: $-\frac{\pi}{2} \le \sin^{-1} u \le \frac{\pi}{2}$

$\therefore \boxed{-\frac{\pi}{2} \le y \le \frac{\pi}{2}}$ $(\frac{1}{2})$

d) $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{3-x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{3}}\right)\Big]_0^{\sqrt{3}}$

$= \sin^{-1}(1) - \sin^{-1}(0)$

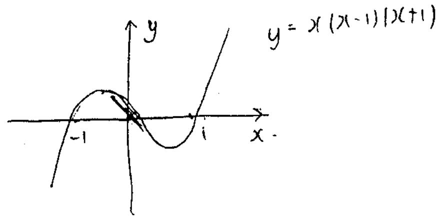
$= \frac{\pi}{2}$

(e) $\dfrac{x}{x^2-1} > 0$ $\boxed{x \neq \pm 1}$

$\therefore \dfrac{x}{(x+1)(x-1)} > 0$

$[\times (x+1)^2 (x-1)^2]$

$\therefore (x+1)(x-1)x > 0$



$\therefore \boxed{-1 < x < 0, x > 1}$

(f) (i) $\because AB = AC$

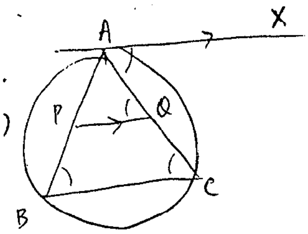
$\therefore \angle ABC = \angle ACB$ (base angles of isos. Δ)

$X\hat{A}C = A\hat{B}C$ (alternate segment theorem)

$X\hat{A}C = A\hat{Q}P$ (alternate angles)

$\therefore A\hat{Q}P = A\hat{C}B$

$\therefore PQ \parallel BC$ (corresponding angles equal)



$\therefore A\hat{P}Q = A\hat{Q}P$

$\therefore AP = AQ$ (isos Δ)

(ii) $BC \parallel PQ$ & $PQ \parallel AX$

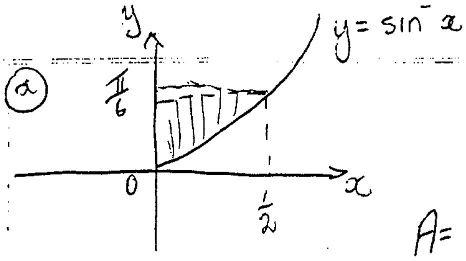
$\therefore BC \parallel AX$

(iii) $\angle A\hat{P}Q = A\hat{C}B$ (from (i))

$\therefore$ exterior angle equals opposite interior angle

$\therefore PQCB$ is cyclic quad.

③ (a)



$\sin^{-1}\frac{1}{2} = 30^\circ = \frac{\pi}{6}$

If $y = \sin^{-1} x$ then $\sin y = x$

$$A = -\int_0^{\frac{\pi}{6}} -\sin y \, dy$$

$$= -\cos y \Big]_0^{\frac{\pi}{6}}$$

$$= -\cos\frac{\pi}{6} + \cos 0$$

$$= -\frac{\sqrt{3}}{2} + 1$$

$$= 1 - \frac{\sqrt{3}}{2}.$$

Now area rectangle is $\frac{1}{2} \times \frac{\pi}{6} = \frac{\pi}{12}$. (exact)

area required is $\frac{\pi}{12} - \left(1 - \frac{\sqrt{3}}{2}\right) = \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}$ u^2

(b) In $\triangle BAQ$, $\tan 22^\circ = \frac{h}{AQ}$

$h = AQ \tan 22^\circ$

$AQ = \frac{h}{\tan 22^\circ} = h\cot 22^\circ$

④

In $\triangle BAP$ $\tan 43^\circ = \frac{h}{AP}$

$h = AP \tan 43^\circ$

$AP = \frac{h}{\tan 43^\circ} = h\cot 43^\circ$

④

Now In $\triangle APQ$, $AP^2 + PQ^2 = AQ^2$

$AP^2 - AQ^2 = -160000$

$h^2\cot^2 43^\circ - h^2\cot^2 22^\circ = -160000$

$$h^2 = \frac{-160000}{\cot^2 43^\circ - \cot^2 22^\circ} = \frac{160000}{\cot^2 22^\circ - \cot^2 43^\circ}$$

3 (c) $\int \sec^2 x \tan^2 x \, dx$

Let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x \, . \, dx$

$= \int u^2 . du$

$= \frac{u^3}{3} + c$

$= \frac{\tan^3 x}{3} + c$

4

Q4 (a) (i) $y = px - ap^2$ — ①

$y = qx - aq^2$ — ②

$0 = (p-q)x - a(p^2-q^2)$

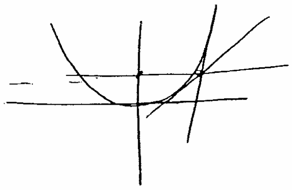
$x = \frac{a(p^2-q^2)}{p-q}$

$\boxed{x = a(p+q)}$ ↗ $y = pa(p+q) - ap^2$

$= ap^2 + apq - ap^2$

$\boxed{y = apq}$

$\therefore \boxed{A \text{ is } (a(p+q), apq)}$ 3



(ii) If $y = a$ then $apq = a$

$\boxed{pq = 1}$ 1

(iii) Chord $\boxed{y = \frac{1}{2}(p+q)x - apq}$

Show $\frac{y - ap^2}{x - 2ap} = \frac{aq^2 - ap^2}{2aq - 2ap} = \frac{q+p}{2}$

$\therefore y - ap^2 = \left(\frac{p+q}{2}\right)x - \frac{2ap}{2}(p+q)$

$y - ap^2 = \frac{p+q}{2}x - ap^2 - apq$

$\boxed{y = \frac{p+q}{2}x - apq}$ 2

now if $x = 0$ $y = -apq$ 2

$\boxed{y = -a}$ because $pq = 1$

$-8 - 8 + 3$

(b) $y = x^3 - 2x^2 + 3$

$\frac{dy}{dx} = 3x^2 - 4x$ ✓✓

$m = 3\times4 - 4\times2$ ✓ 1.

$\underline{m = 4}$

$\frac{y-3}{x-2} = 4$

$y - 3 = 4x - 8$

$\boxed{y = 4x - 5}$

2

Solve $x^3 - 2x^2 + 3 = 4x - 5$

$x^3 - 2x^2 - 4x - 8 = 0$ ✓✓

roots are $2, 2, \alpha$

$2 + 2 + \alpha = 2$

$\alpha = -2$

$\therefore (-2, -13)$ 2

5(a) Test for $n=1$, $S_1 = 3^3 + 2^3$

$= 27 + 8$ ✓

$= 35$ which is divisible by 5.

Assume true for $n = k$, ie $S_k = 3^{3k} + 2^{k+2}$ ✓

$= 5P$ where $P \in J$.

Now test for $n = k+1$, ie $S_{k+1} = 3^{3k+3} + 2^{k+3}$

$= 5Q$ where $Q \in J$.

$S_{k+1} = 27(5P - 2^{k+2}) + 2.2^{k+2}$

$= 5.27P - (27-2)2^{k+2}$

$= 5\{27P - 5.2^{k+2}\}$ ✓

$= 5Q$.

$\therefore$ True for $n = k+1$ if true for $n = k$.

Now true for $n = 1$ so true for $n = 2$ and so on for all integer n. ✓

(b) $f(x) = x^3 - 7$, $f'(x) = 3x^2$ ✓

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ ✓

$= 2 - \dfrac{8-7}{3.4}$

$= {}^{23}/_{12}$. ✓

(c) $\dfrac{v^2}{2} = 45 - 6x - 3x^2$

$\ddot{x} = \dfrac{d}{dx}\left(\dfrac{v^2}{2}\right)$

$= -6 - 6x$

$= -6(x+1)$ ✓

$= -(\sqrt{6})^2 X$ where $X = x + 1$

$\therefore$ Motion is SHM with centre of motion $= -1$.

$n = \sqrt{6}$ so period $= {}^{2\pi}/_{\sqrt{6}} = {}^{\sqrt{6}\pi}/_{3}$ ✓

$v^2 = -6(x^2 + 2x + 1) + 90 + 6$

$= 96 - 6(x+1)^2$

$= 6\{4^2 - (x+1)^2\}$ ✓

Amplitude = 4

5(d)(i) $RHS = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} \times \dfrac{\cos^2\theta}{\cos^2\theta}$

$= \dfrac{\sin^2\theta - \cos^2\theta}{\sin^2\theta + \cos^2\theta}$

$= \cos 2\theta$ ✓

$= LHS.$

(ii) If $\theta = \pi/8$, $\cos 2\theta = \dfrac{1}{\sqrt{2}} = \dfrac{1-x^2}{1+x^2}$

$1 + x^2 = \sqrt{2} - \sqrt{2}\,x^2$

$x^2(1+\sqrt{2}) = \sqrt{2} - 1$

$x^2 = \dfrac{\sqrt{2}-1}{\sqrt{2}+1} \times \dfrac{\sqrt{2}-1}{\sqrt{2}-1}$

$= \dfrac{(\sqrt{2}-1)^2}{2-1}$ ✓

$x = \sqrt{2} - 1$ as $\tan \pi/8$ is in 1st quadrant.

Question 6

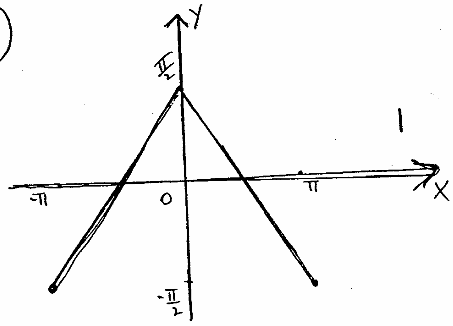
(a)

$y = \sin^{-1}(\cos x)$

(i)
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\cos^2 x}} \cdot -\sin x$$
$$= \frac{-\sin x}{\sqrt{\sin^2 x}} \qquad 2$$
$$= -\frac{\sin x}{|\sin x|}$$
$$= -1 \text{ for } 0 < x < \pi$$
$$= 1 \text{ for } -\pi < x < 0$$

(ii) $y = \sin^{-1}[\cos \pi]$
$$= \sin^{-1}[-1] \qquad 1$$
$$= -\sin^{-1}[1]$$
$$= -\frac{\pi}{2}$$

(iii)



1

(b) $\dot{x} = V\cos\theta = 35\cos 0 = 35$

$\dot{y} = 35\sin 0 - 10t = -10t$

$y = 1.8 - 5t^2$

$x = Vt\cos\theta = 35t$

(i) Strikes ground when $y = 0$

$\therefore 0 = 1.8 - 5t^2$

$t = 3/5$ sec. $\quad$ 2

(ii) $x = 35 \times \frac{3}{5} = 21$ m $\quad$ 2

(iii) When $x = 14$, $\quad 14 = 35t$

ie $t = 2/5$

When $t = 2/5$, $y = 1.8 - 5\left(\frac{2}{5}\right)^2$ $\quad$ 2

ie $y = 1$ m

$\therefore$ Clears net by $1 - 0.95$ m $= 5$ cm.

(c)

(i) $\frac{d}{dx}(xe^x) = xe^x + 1.e^x = e^x(x+1)$ ½

(ii) $\frac{d}{dx} xe^x - e^x = xe^x$

ie $\frac{d}{dx}[xe^x - e^x] = xe^x$ ½

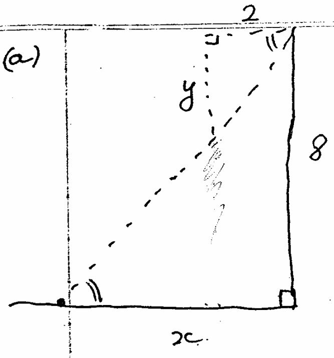
$\Rightarrow \int xe^x\,dx = xe^x - e^x$ ½ $\quad$ 1

$\therefore \int_0^1 xe^x\,dx = \left[xe^x - e^x\right]_0^1$ ½

$= [(e - e) - (0 - e^0)]$

$= 1$

7 (a)



(i) Triangles are similar

$\therefore \quad \dfrac{x}{2} = \dfrac{8}{y}$ 2

$x = \dfrac{16}{y}$

(ii) $\dfrac{dx}{dt} = \dfrac{dx}{dy} \cdot \dfrac{dy}{dt}$ 2

$= -\dfrac{16}{y^2} \cdot 10$

When $y = 6$, $\dfrac{dx}{dt} = \dfrac{-16}{36} \cdot 10$

$= -\dfrac{40}{9}$ m/s.

(iii) $\dfrac{dy}{dt} = \dfrac{dx}{dt}$ $\qquad \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = \dfrac{dx}{dt}$ $\qquad -\dfrac{16}{y^2} = -1$ (speed)

$y = 4$. 2

(b) (i) $f(x) = (e^x - 1)\ln x$

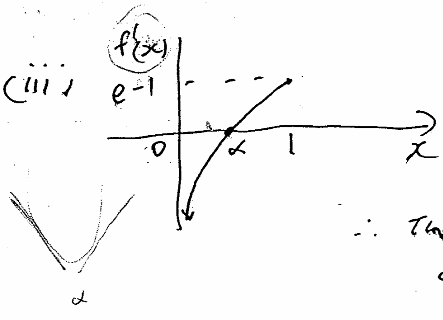
$f'(x) = \dfrac{e^x - 1}{x} + e^x \ln x$. 2

$f'(1) = e - 1 + 0 = e - 1$

(ii) As $x \to 0$, $f'(x) \to -\infty$ 1

since $\ln x \to -\infty$ as $x \to 0$

(iii)



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$f'(x) = 0$ for $0 < x < 1$

Let this root be α.

For $x < \alpha$ $\quad f'(x) < 0$

$x > \alpha$ $\quad f'(x) > 0$ 2

$\therefore$ The stationary point is a local minimum.